Solitons and coherent structures in optics and superconductivity

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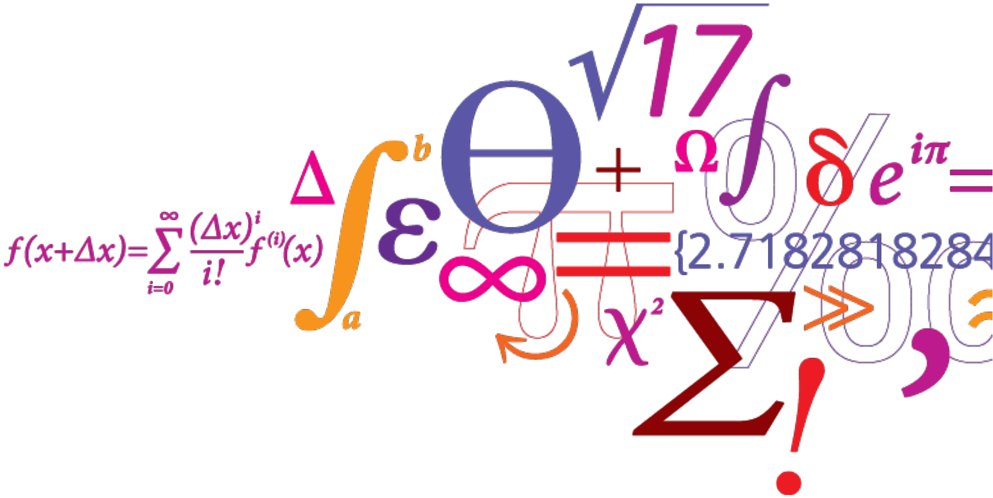

Collaborators

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Introduction

- **Lagrangian variation analysis of perturbed solitons. Exemplified by the Nonlinear Schrödinger (NLS) equation.**

- **Vortices in type II superconductors modelled by the Ginzburg-Landau equations. Application to pinning of vortices on impurities. Dynamics of vortices in superconductors with pinning sites.**

The Nonlinear Schrödinger (NLS) Equation

NLS equation.

$$iu_z + u_{tt} + 2|u|^2 u = 0$$

Simple soliton solution.

$$u(z,t) = \rho(z,t)e^{i\varphi(z,t)} = \eta \operatorname{sech}\left(\eta\theta\right)\exp(i(\xi\theta + \sigma))$$

$$\theta = t - t_0 - 2\xi z \qquad \sigma = \left(\eta^2 + \xi^2\right)z - \sigma_0$$

The perturbed NLS equation.

$$iu_z + u_{tt} + 2|u|^2 u = \varepsilon R(u)$$

Damping and laser power input.

$$\varepsilon R(u) = -i\Gamma u + \frac{ig_0}{1 + \left(\frac{P}{P_s}\right)}\left(u + w\frac{\partial^2 u}{\partial t^2}\right)$$

$$P = \int_{-\infty}^{\infty} \left|u(z,t)\right|^2 dt$$

Ref.: Amplification of ultrashort solitons in erbium-doped fiber amplifiers, (G. P. Agrawal). IEEE Photon. Technol. Lett. 2, (1990) pp. 875–877

The Perturbed NLS Equation

Further perturbation terms includes:

1) **Higher order dispersions**
2) **Raman gain.**

Perturbation analysis. Collective coordinate approach.

Collective coordinate approach. Variational method invoking generalized forces associated with each collective coordinate.

$$\left\{\eta, \xi, t_0, \sigma_0\right\} = \{y_1(z), y_2(z), y_3(z), y_4(z)\}$$

Ref.: *Ring laser configuration studied by collective coordinates, (Caputo, Flytzanis, Sørensen)*. J. Opt. Soc. Am. B, Vol. 12 (1) (1995), 139-145

Lagrangian, Momentum, Hamiltonian

The Lagrangian density:

$$\mathscr{L} = \frac{i}{2}\left(u^* u_z - u_z^* u\right) - \left|u_t\right|^2 + \left|u\right|^4$$

The momentum density:

$$p = \frac{\partial \mathscr{L}}{\partial u_z} = +\frac{i}{2}u^* \qquad p^* = \frac{\partial \mathscr{L}}{\partial u_z^*} = -\frac{i}{2}u$$

The Hamiltonian:

$$\mathscr{H} = u_z p + u_z^* p^* - \mathscr{L}$$

The Collective Coordinate Approach

The total energy:

$$E = \int_{-\infty}^{\infty} \mathscr{H}(z,t)dt = 2\eta\xi^2 - \frac{2}{3}\eta^3$$

The total Lagrangian:

$$L = \int_{-\infty}^{\infty} \mathscr{L}\left(u, u^*, u_z, u_z^*, u_t, u_t^*\right)dt = L(y_i(z),\ y_i'(z)) \qquad i = 1,2,3,4$$

The Euler Lagrange equations for the collective coordinates reads:

$$\frac{\partial L}{\partial y_i} - \frac{d}{dz}\left(\frac{\partial L}{\partial y_i'}\right) = \varepsilon \int_{-\infty}^{\infty} R \frac{\partial u^*}{\partial y_i} dt + c.c.$$

The Collective Coordinate Approach

Note: $u = u(z, t; y_i(z))$ $\qquad \dfrac{du}{dz} = u_z(z, t; y_i(z), y_i'(z))$

Differentiation rules:

$$\frac{\partial L}{\partial y_i} = \int_{-\infty}^{\infty} \left\{ \frac{\partial \mathscr{L}}{\partial u}\frac{\partial u}{\partial y_i} + \frac{\partial \mathscr{L}}{\partial u_z}\frac{\partial u_z}{\partial y_i} + \frac{\partial \mathscr{L}}{\partial u_t}\frac{\partial u_t}{\partial y_i} \right\} dt + c.c. \qquad i = 1,2,3,4$$

$$\frac{\partial L}{\partial y_i'} = \int_{-\infty}^{\infty} \left\{ \frac{\partial \mathscr{L}}{\partial u_z}\frac{\partial u_z}{\partial y_i'} \right\} dt + c.c. = \int_{-\infty}^{\infty} \left\{ \frac{\partial \mathscr{L}}{\partial u_z}\frac{\partial u}{\partial y_i} \right\} dt + c.c.$$

Partial integration

$$\int_{-\infty}^{\infty} \frac{\partial \mathscr{L}}{\partial u_t}\frac{\partial u_t}{\partial y_i} dt = \int_{-\infty}^{\infty} \frac{\partial \mathscr{L}}{\partial u_t}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y_i}\right) dt = \left[\frac{\partial \mathscr{L}}{\partial u_t}\left(\frac{\partial u}{\partial y_i}\right)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial u_t}\right)\frac{\partial u}{\partial y_i} dt$$

The Collective Coordinate Approach

We need to calculate:

$$\frac{\partial}{\partial z}\left(\frac{\partial L}{\partial y_i'}\right) = \int_{-\infty}^{\infty} \frac{\partial}{\partial z}\left\{\frac{\partial \mathscr{L}}{\partial u_z}\frac{\partial u}{\partial y_i}\right\} dt = \int_{-\infty}^{\infty} \left\{\frac{\partial}{\partial z}\left(\frac{\partial \mathscr{L}}{\partial u_z}\right)\frac{\partial u}{\partial y_i} + \frac{\partial \mathscr{L}}{\partial u_z}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y_i}\right)\right\} dt$$

$$i = 1,2,3,4$$

The Euler Lagrange equations for the collective coordinates reads:

$$\frac{\partial L}{\partial y_i} - \frac{d}{dz}\left(\frac{\partial L}{\partial y_i'}\right) = \int_{-\infty}^{\infty} \left\{\frac{\partial \mathscr{L}}{\partial u}\frac{\partial u}{\partial y_i} + \frac{\partial \mathscr{L}}{\partial u_z}\frac{\partial u_z}{\partial y_i} - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial u_t}\right)\frac{\partial u}{\partial y_i}\right\} dt$$

$$- \int_{-\infty}^{\infty} \left\{\frac{\partial}{\partial z}\left(\frac{\partial \mathscr{L}}{\partial u_z}\right)\frac{\partial u}{\partial y_i} + \frac{\partial \mathscr{L}}{\partial u_z}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y_i}\right)\right\} dt + c.c$$

The Collective Coordinate Approach

From this we obtain the dynamical equations for the collective coordinates:

$$\frac{\partial L}{\partial y_i} - \frac{d}{dz}\left(\frac{\partial L}{\partial y_i'}\right) = \int_{-\infty}^{\infty} \left\{ \frac{\partial \mathscr{L}}{\partial u} - \frac{\partial}{\partial z}\left(\frac{\partial \mathscr{L}}{\partial u_z}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial u_t}\right) \right\} \frac{\partial u}{\partial y_i} dt + c\,.\,c\,. \qquad i = 1,2,3,4$$

$$= \varepsilon \int_{-\infty}^{\infty} R^* \frac{\partial u}{\partial y_i}\, dt + c\,.\,c\,.$$

Example: The ring laser configuration

Higher order dispersion and nonlinearities have been included in the model of the fiber ring laser

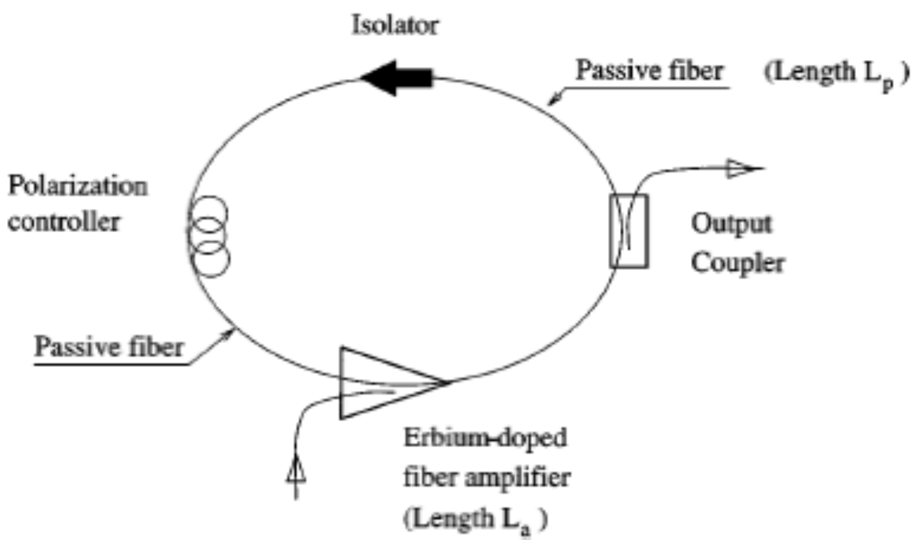


Fig. 1. Fiber laser ring configuration.

Ref.: *Ring laser configuration studied by collective coordinates, (Caputo, Flytzanis, Sørensen)*. **J. Opt. Soc. Am. B, Vol. 12 (1) (1995), 139-145**

Example: The ring laser configuration

The dynamical equations for the collective coordinates

$$\frac{d\eta}{dt} = -2\Gamma\eta + \frac{2g_0\eta}{1 + 2\eta^2/p_s}$$

$$\frac{d\xi}{dt} = 0$$

$$\frac{d\sigma}{dt} = \eta^2 + \xi^2$$

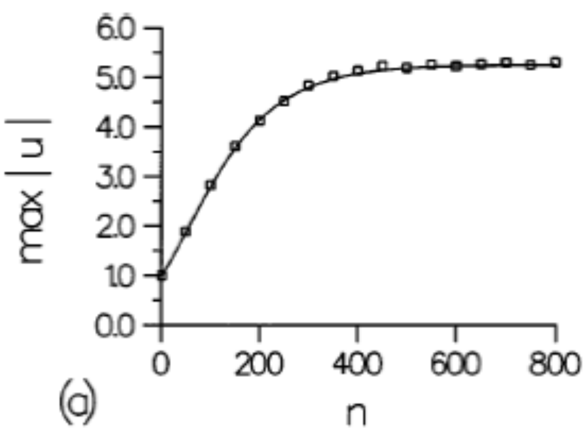
$$\frac{d\theta}{dt} = -2\xi$$

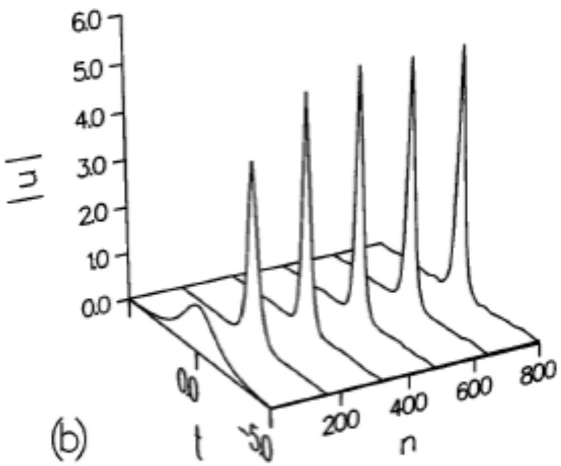
To each collective coordinate is an associated generalized force, resulting from the perturbations.

Ref.: Ring laser configuration studied by collective coordinates, (Caputo, Flytzanis, Sørensen). J. Opt. Soc. Am. B, Vol. 12 (1) (1995), 139-145

Numerical example

Numerical simulations of the full PDE compared to the dynamical equations for the collective coordinates





The Time Dependent Ginzburg–Landau Model

The Dynamics of Magnetic Vortices in Type II Superconductors Studied by the Time Dependent Ginzburg-Landau Model

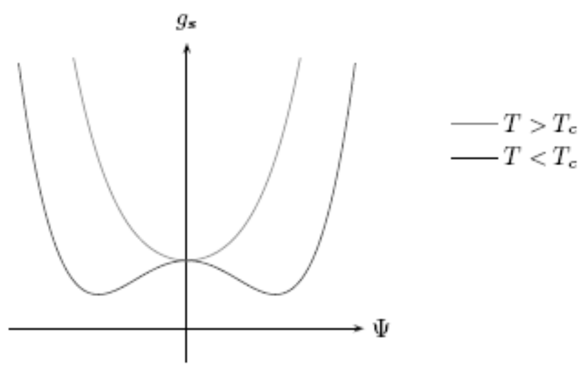


Figure 1.1: The Gibbs function g_s for the normal (gray) and superconducting (black) state.

$$G_s = G_n - \alpha(T)|\Psi|^2 + \frac{\beta}{2}|\Psi|^4$$

$$\alpha(T) = \alpha(0)(1 - T/T_c)$$

Ref.: On the Theory of Superconductivity. V.L. Ginzburg and L.D. Landau. JZh. Eksp. Teo. Fiz., 20, p1064 (1959)

The Time Dependent Ginzburg–Landau Model

The Ginzurg-Landau equation in nondimensional coordinates and after the coulomb gauge transformation reads

$$\frac{\partial \psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + A\right)^2 \psi + \psi - \left|\psi\right|^2 \psi \qquad \text{in} \quad \Omega$$

$$\sigma \frac{\partial A}{\partial t} = \frac{1}{2i\kappa}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) - \left|\Psi\right|^2 A - \nabla \times \nabla \times A$$

$\psi(x, y, t)$ **is the order-parameter for the superconducting Cooper pair condensate**

$A(x, y, t)$ **is the magnetic vector potential**

$\kappa = \lambda / \xi$ **is the Ginzburg-Landau parameter**

Ref.: ***Magnetic Flux Lines in Complex Geometry Type-II Superconductors Studied by the Time Dependent Ginzburg-Landau Equation.*** **T.S. Alstrøm, M.P.Sørensen, N.F. Pedersen and S. Madsen. Acta Appl Math, 2010, 1-12.**

The Time Dependent Ginzburg–Landau Model

The superconducting current is $$J_s = \frac{1}{2i\kappa}(\psi^* \nabla \psi - \psi \nabla \psi^*) - |\psi|^2 A$$

The normal current is $$J_n = -\sigma \partial_t A$$

We obtain the boundary conditions on $\partial\Omega$

$$\left(\frac{i}{\kappa}\nabla + A\right)\psi \bullet n = 0 \qquad \nabla \times A = B_{ext} \qquad J_n \bullet n = -\sigma \partial_t A \bullet n = 0$$

Ref.: *Self-consistent Ginzburg-Landau theory for transport currents in superconductors.* **M. Ögren, M.P.Sørensen, N.F. Pedersen. Physica C, 479 (2012) 157-159.**

Energy densities

The superconducting energy is

$$H_{sup} = \frac{1}{\kappa^2}\left|\nabla\psi\right|^2 - \left|\psi\right|^2 + \frac{1}{2}\left|\psi\right|^4$$

The magnetic energy is

$$H_{mag} = \left(B_{ext} - \nabla \times A\right)^2$$

The interaction energy is

$$H_{int} = \frac{i}{\kappa}A\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) + \left|A\right|^2\left|\psi\right|^2$$

The total energy is

$$H = \int_\Omega \left(H_{sup} + H_{mag} + H_{int}\right) d\Omega$$

Ref.: Self-consistent Ginzburg-Landau theory for transport currents in superconductors. M. Ögren, M.P.Sørensen, N.F. Pedersen. Physica C, 479 (2012) 157-159.

Numerical method

Finite element programme by COMSOL Multiphysics, using quadratic Lagrange elements.

Complex shapes of the superconductors

The complex valued function ψ is split into its real and imaginary parts. Vector A has two components. Total of four coupled PDEs.

$$d_a \frac{\partial u}{\partial t} + \nabla \cdot \Gamma = F \qquad \text{in} \qquad \Omega$$

Boundary conditions $\quad -n \cdot \Gamma = G \qquad$ on $\quad \partial\Omega$

Ref.: The Ginzburg Landau Equation Solved by the Finite Element Method . T.S. Larsen, et al.. Proc. of the Nordic COMSOL Conference, Nov. 1-2, 2006, Kgs. Lyngby, Denmark. Ed. L. Gregersen, pp 75-78, 2007.

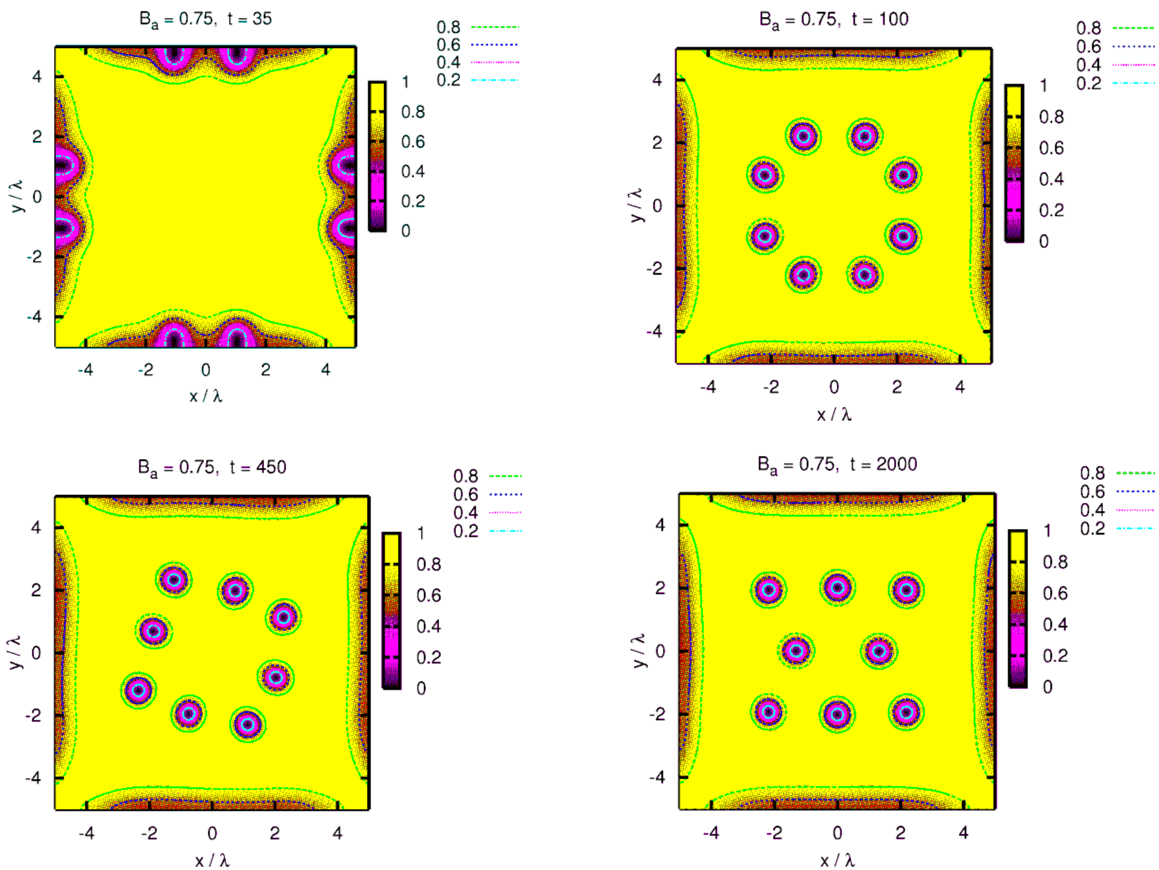
Numerical result

Square superconductor

$$\left|\psi(x,y,t)\right|$$

$$B_{ext} = 0.75$$

$$\kappa = 4$$



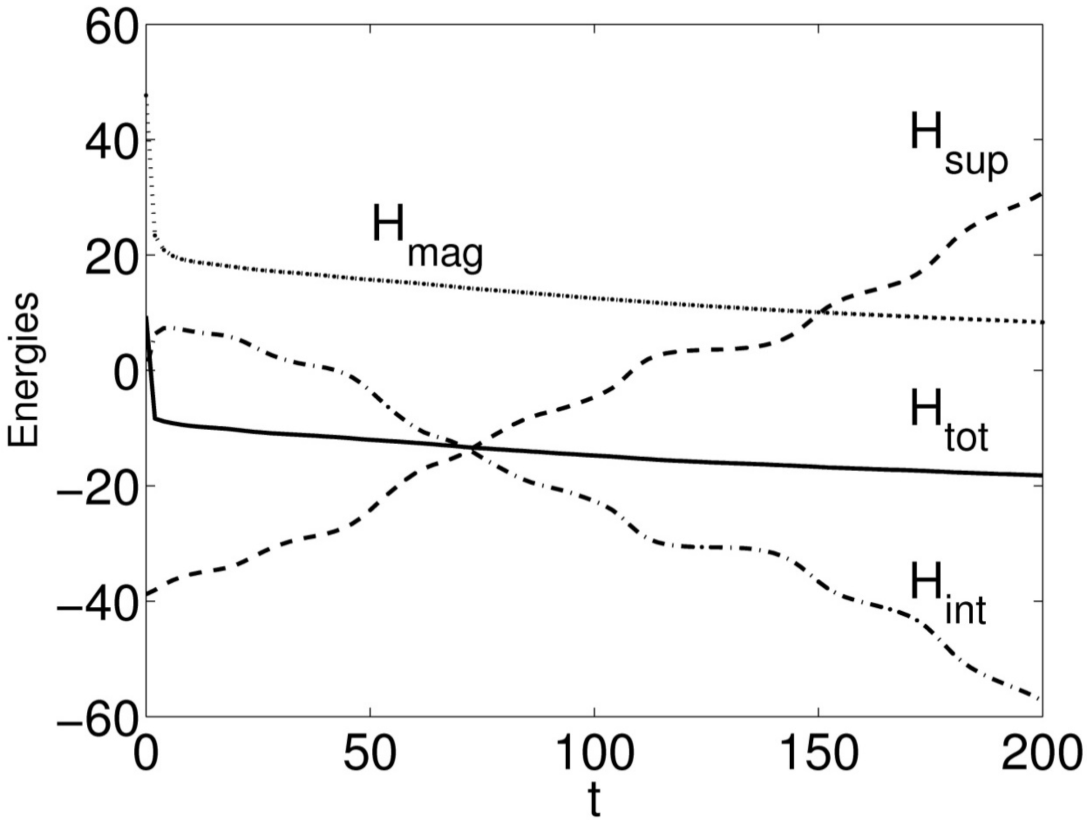
Numerical result

Circular superconductor with an indent

$\left|\psi(x,y,t)\right|$

$B_{ext} = 0.8$

$\kappa = 4$

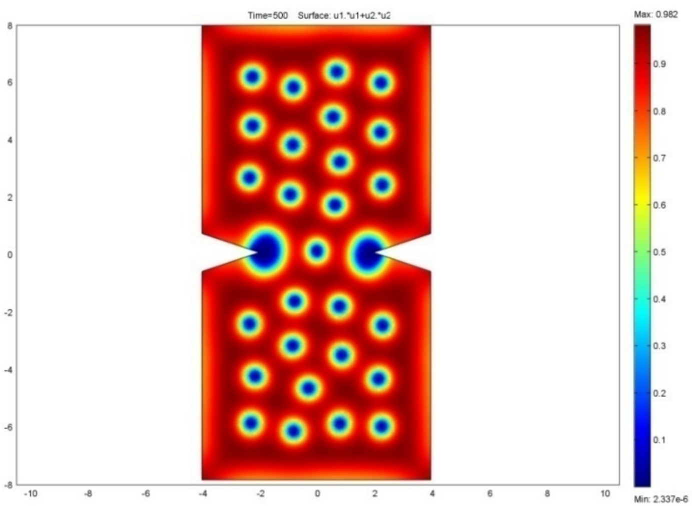


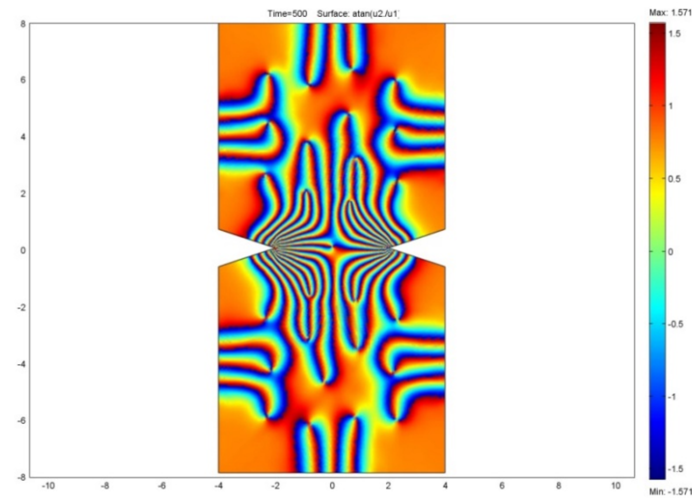
Numerical result

Josephson weak link

$$\left|\psi(x, y, t)\right|$$

$$\varphi = \tan^{-1}\left(\frac{Im(\psi)}{Re(\psi)}\right)$$





$B_{ext} = 0.8$ $\kappa = 4$ $t = 500$

Pinning sites for vortices

Phenomenological model for the Gibbs energy

$$G_s = G_n - \alpha_0(r)\left(1 - \frac{T}{T_c}\right)\left|\psi\right|^2 + \frac{\beta}{2}\left|\psi\right|^4$$

Ref.: The dynamics of magnetic vortices in type II superconductors with pinning sites studied by the time dependent Ginzburg–Landau model. M. P. Sørensen a , N. F. Pedersen and M. Ögren, Physica C: Superconductivity and its applications 533 (2017) 40–43

How to make a permanent type II high-Tc magnet by pinning?

Place the type II superconductor with pinning sites in an external magnetic field, which is raised from zero to above the first critical magnetic field strength.

The superconductor is now magnitized

Remove the magnetic field.

For sufficiently strong pinning sites, vortices will remain bounded on the pinning sites, leaving the superconductor a permanent magnet.

Pinning sites for vortices

Phenomenological model in dimension-less units

$$\frac{\partial \psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + A\right)^2 \psi + f(r)\,\psi - \left|\psi\right|^2 \psi$$

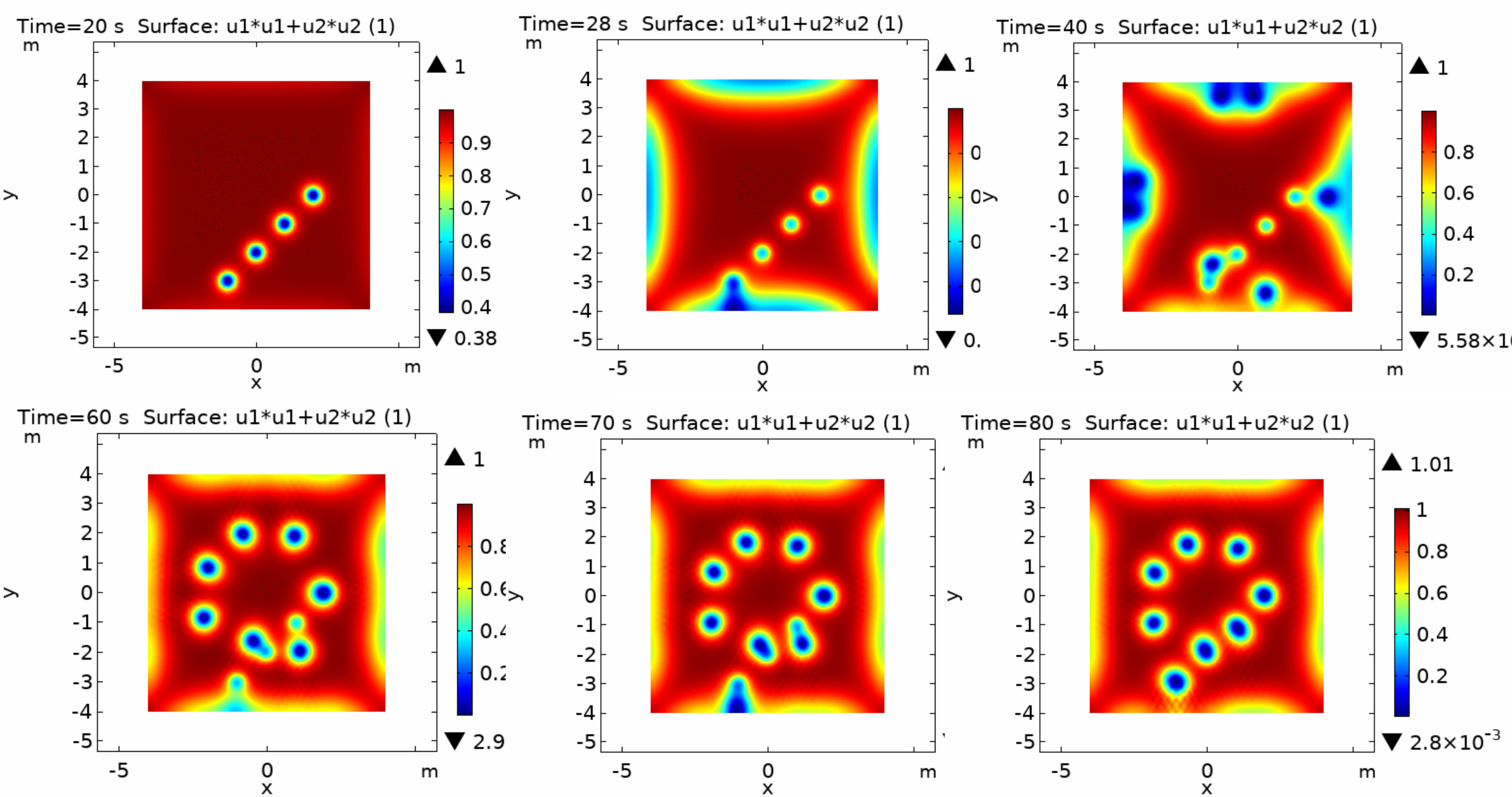
$$f(r) = \prod_{k=1}^{N} f_k(r)$$

and with

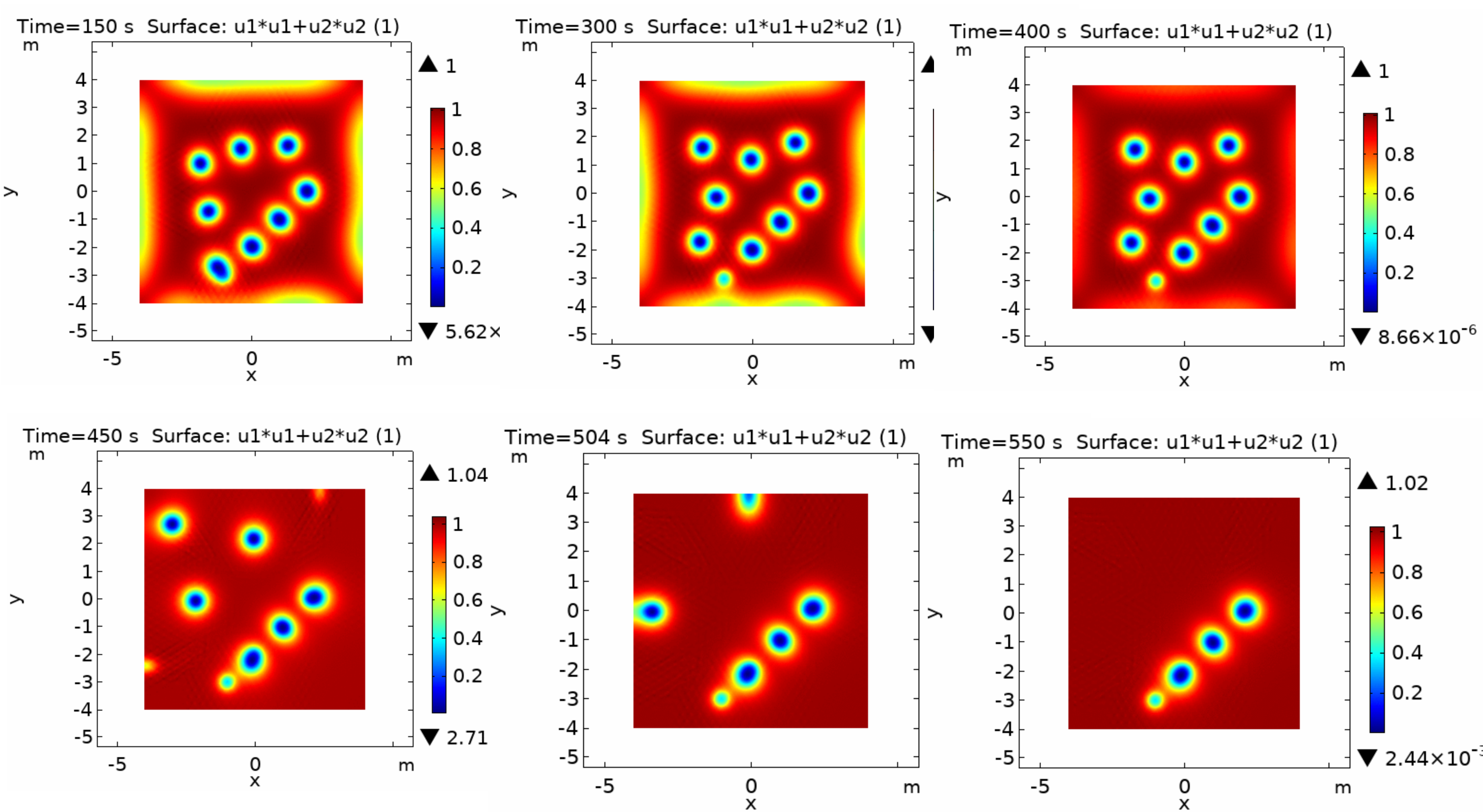
$$f_k(r) = \tanh\left(\frac{\left|r - r_{0k}\right| - R_k}{w_k}\right)$$

Ref.: ***The dynamics of magnetic vortices in type II superconductors with pinning sites studied by the time dependent Ginzburg–Landau model.*** **M. P. Sørensen a , N. F. Pedersen and M. Ögren, Physica C: Superconductivity and its applications 533 (2017) 40–43**

Numerical results



Numerical results



Summary

1) Variational approach for the time evolution of the collective coordinates of solitons. Exemplified by the perturbed NLS equation
2) Generalized forces associated with each collective coordinate.
3) Fiber ring laser

1) The time dependent Ginzburg Landau equations implemented in Comsol Multiphysics
2) Dynamics of penetrating magnetic vortices into a type II superconductor
3) Vortex dynamics in presence of defects
4) Permanent type II superconducting magnet