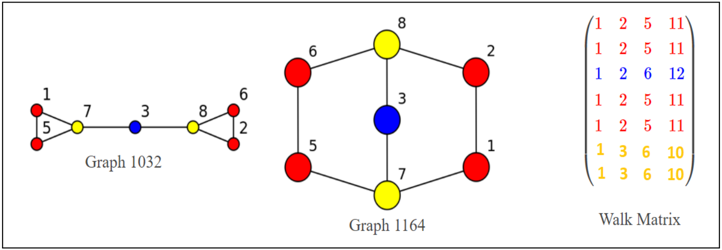
Base Graphs

Canonical Double Cover



University of Malta

http://staff.um.edu.mt/isci1/

Irene Sciriha

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Joint work with Luke Collins

Laboratoire de Mathématiques

2022–Jul03–6

INSA de Rouen, Normandie 2022

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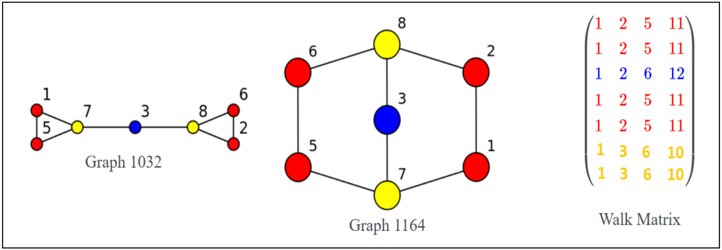

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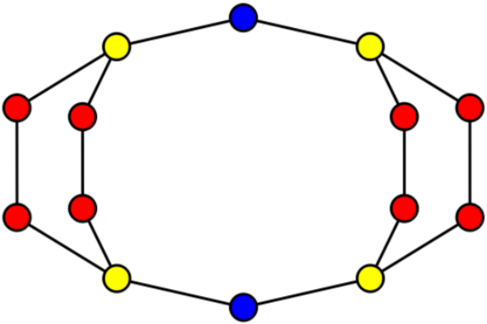
Motivation

Given a bipartite graph, is it the canonical double cover (CDC) of some base graph?



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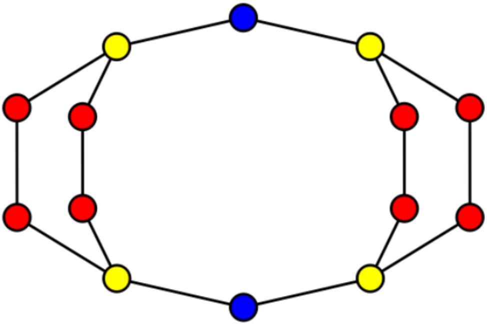

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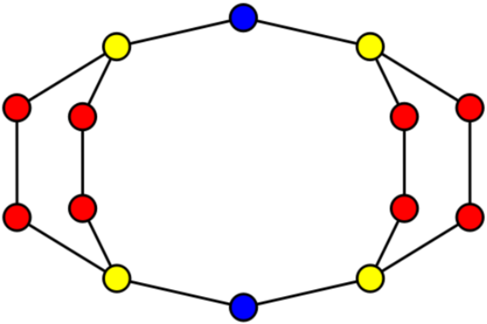

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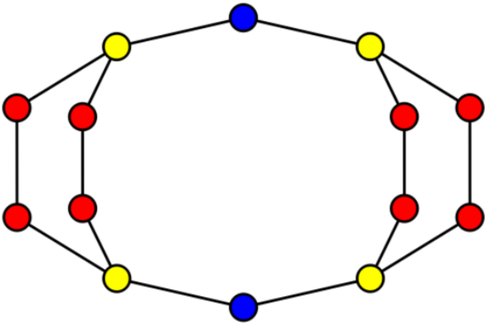

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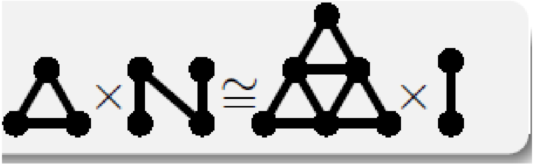
Factorization under direct product

Factorization under direct product

W. Imrich and T. Pisanski (2008): The factorization of a non-bipartite connected graph under direct product is unique.

R. Hammack, W. Imrich and S. Klavzar (2011): For a bipartite connected graph, factorization may not be unique.

D. Witte Morris(2022): Bip Conn Gr has a bipartite factor which may NOT be unique.



The CDC is Bipartite

The canonical double cover (CDC) of a base graph G

is its direct product $G \times K_2$ with $K_2 = a \longleftrightarrow b$.

Edge $(u, a), (w, b)$ is an edge of CDC iff
(u, w) is an edge of G and (a, b) is the edge of K_2.

Also if a graph is a CDC, then its bipartite factor under direct product is NOT necessarily K_2.

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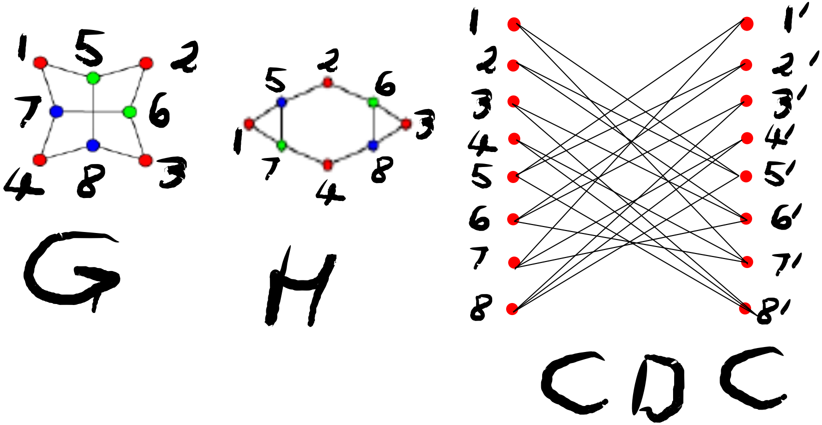
Construction of a CDC



No labelling of H gives the same labelling of its CDC as that of G.

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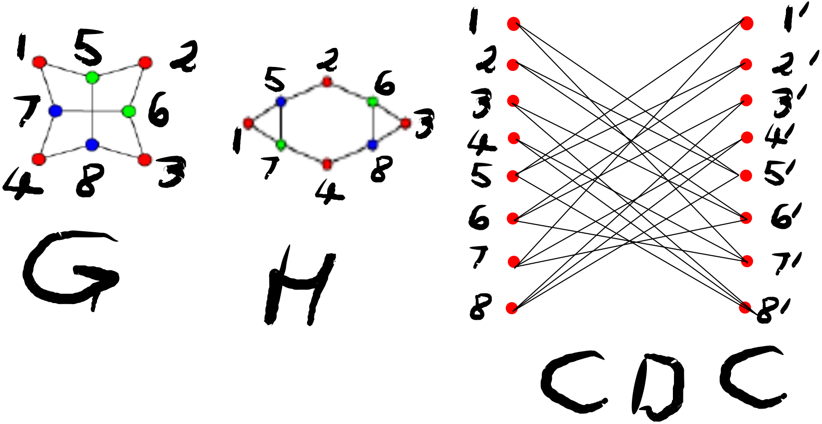

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Inverse Problem

Given a bipartite connected graph B

1. Is it a CDC?
2. Determine base graphs G such that $B = CDC(G)$.

The problem is intractable.

1. The **complexity** of algorithms known to date is factorial, $O(\frac{n}{2})!$
2. **Open Problem:** Is it fixed parameter tractable?

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Existence of a base graph

Starting from the adjacency matrix

$$\mathbf{A}(B) = \left(\begin{array}{c|c} \mathbf{O} & \mathbf{K} \\ \hline \mathbf{K}^\top & \mathbf{O} \end{array}\right), \text{ of a bipartite graph } B,$$

if there exists a graph G with adjacency matrix permutationally equivalent to K and $K^\top$, then $B = CDC(G)$.

Sufficient condition

If there exist permutation matrices P and Q such that $P^\top KQ$ is a real 0–1–symmetric matrix $A(G)$, then B is the CDC of G in Γ_0.

$$\mathbf{A}(CDC) = \left(\begin{array}{c|c} \mathbf{O} & \mathbf{A}(G) \\ \hline \mathbf{A}(G) & \mathbf{O} \end{array}\right),$$

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Unstable base graphs arise from Unexpected Symmetries

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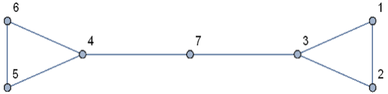

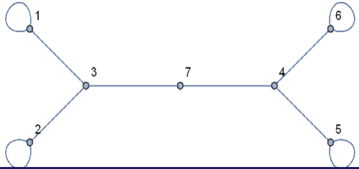

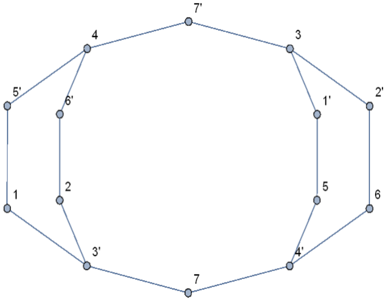

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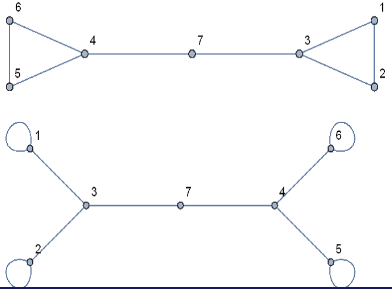

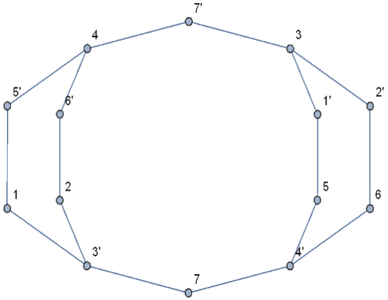

Unstable base graphs

Given a bipartite connected graph with no duplicate vertices

isomorphic to CDC $G \times K_2$,
are there other base graphs H permutationally equivalent to G?

Reduction of the Inverse Problem to

Determining all 2–1 surjective projection mappings from the vertices of the given CDC to the vertices of a base graph, respecting adjacencies.

As 2 vertices of the CDC are identified to obtain G',

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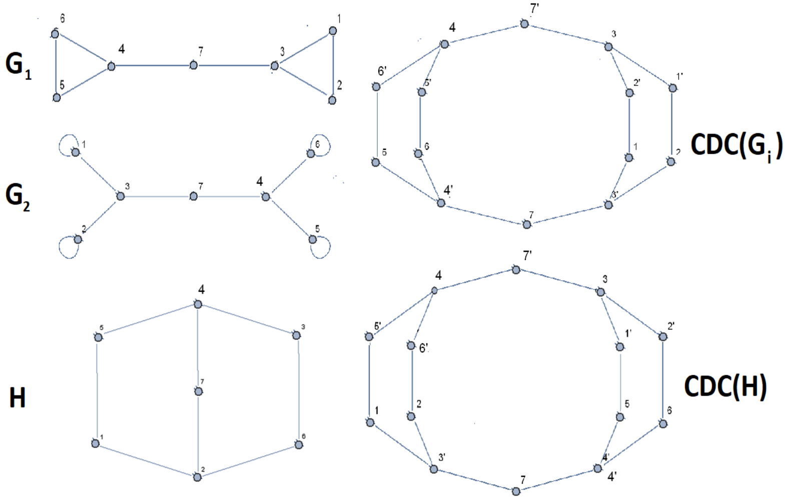

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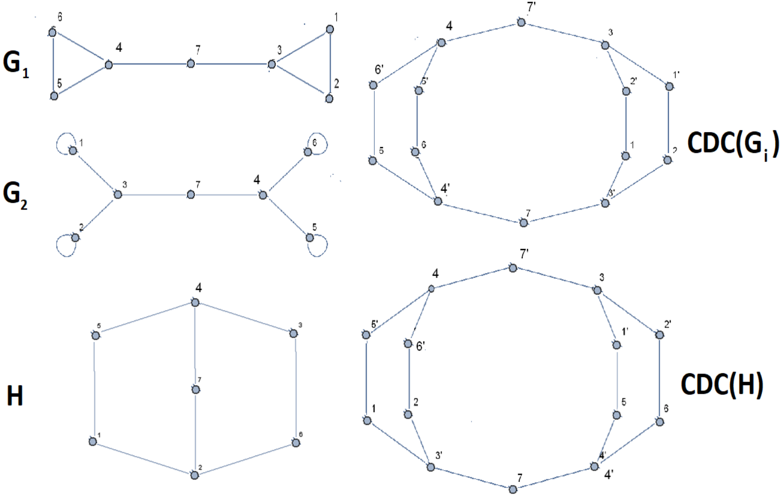

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Permutationally Equivalent Base Graphs

Collins and Sciriha (2020)

Base graphs with isomorphic CDCs have the same k-walk matrix $\mathbf{W}_G(k) = \left(\mathbf{j} \;\middle|\; \mathbf{Aj} \;\middle|\; \mathbf{A}^2\mathbf{j} \;\middle|\; \cdots \;\middle|\; \mathbf{A}^{k-1}\mathbf{j} \right)$. Vertices corresponding to equal rows of $\mathbf{W}_G(k)$ lie in one part of the WALK vertex partition and are assigned one colour in the **WALK COLOURING.**

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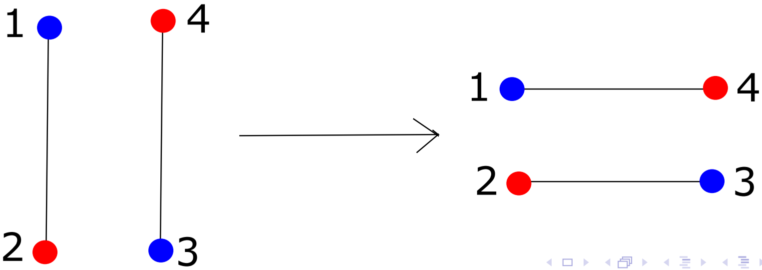
Degree Sequence of Base Graphs

Given by the 2nd column of $\mathbf{W}_G(k)$

- Two graphs with isomorphic CDCs have the same degree sequence.
- Graphs have the same degree sequence iff they are Ryser–switch transformable.

A Ryser–Switch on two disjoint edges



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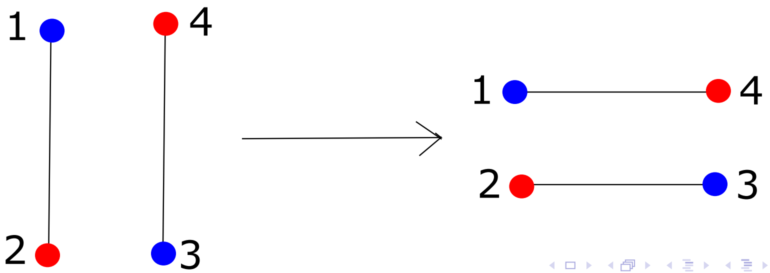

Construction of an unstable base graphs from a stable one

Theorem

Let G and H be graphs having isomorphic CDCs. Then H is isomorphic to a graph obtained from G by RS of disjoint edges each with two common but distinct walk–coloured end–vertices.

Base graphs G and H have the same degree sequence.

So there exist a sequence of Ryser switches involving pairs of disjoint edges $\{u, v\} and \{x, y\}$ in G with the same walk–colouring that are transformed to $\{u, x\} and \{v, y\}$ in H,
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Equivalent to permuting the rows and columns of G differently.

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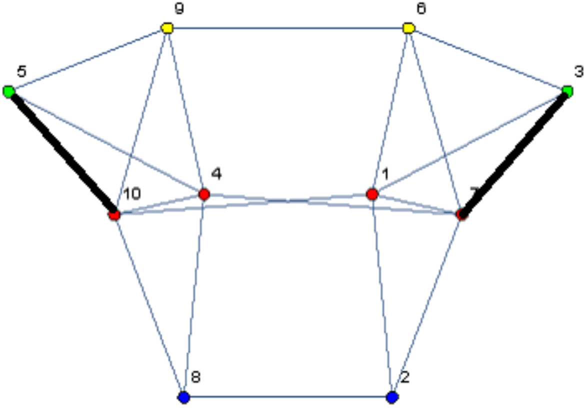

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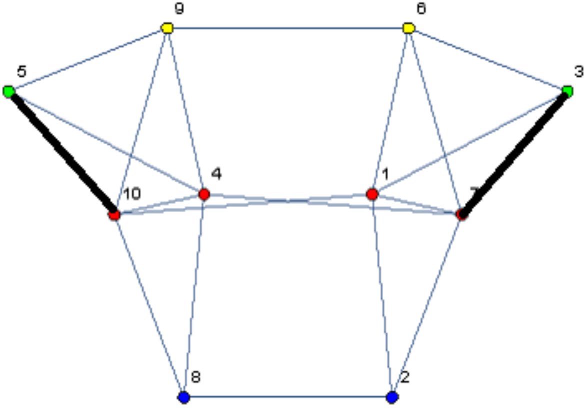

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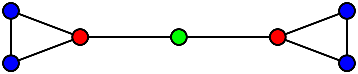

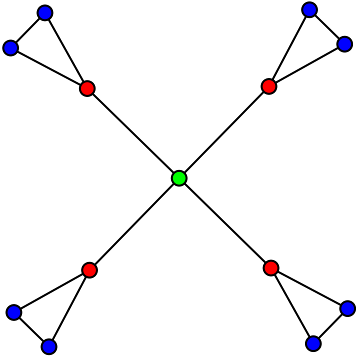

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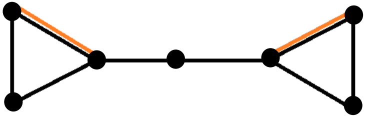

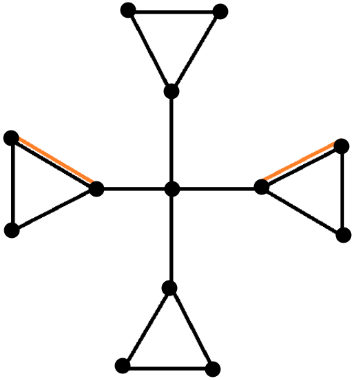

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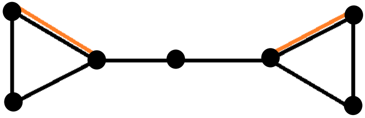

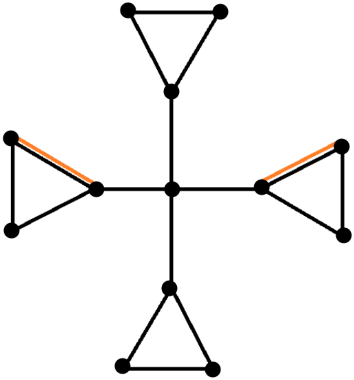

Website

https://maths.mt/walks/same-cdc/