Nonlinear normal modes in a network with cubic couplings

Imene KHAMES

Jean-Guy CAPUTO, Arnaud KNIPPEL

Laboratoire de Mathématiques Appliquées du Havre, Université Le Havre Normandie.

Ondes non-linéaires et réseaux : conférence en l'honneur des 60 ans de Jean-Guy Caputo

Outline

1. Network wave equation
2. On-site nonlinearity: nonlinear normal modes
3. Bivalent and trivalent graphs
4. Intersite nonlinearity: nonlinear normal modes

Plan

1. Network wave equation
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Discrete Gradient

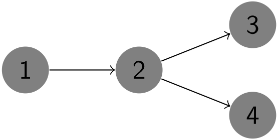
A graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ with vertex set $\mathcal{V}$ of cardinality N and edge set $\mathcal{E}$ of cardinality M.

Incidence matrix $Q \in \mathcal{M}_{N,M}(\mathbb{Z})$

$$Q_{je} = \begin{cases} -1 & \text{if branch } e \text{ starts from node } j, \\ 1 & \text{if branch } e \text{ finishes at node } j, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Gradient of graph : $\quad \nabla = Q^T$.

$$\nabla f = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} f_2 - f_1 \\ f_3 - f_2 \\ f_4 - f_2 \end{pmatrix}$$



Paw graph.

Generalize conservation laws $\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$.

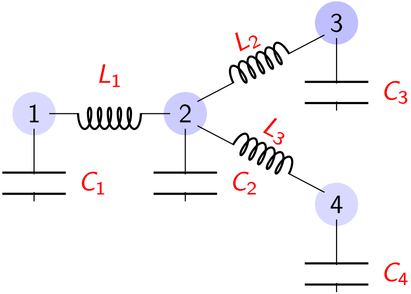
Array of inductances and capacities

$$C\frac{\mathrm{d}v}{\mathrm{d}t} + \nabla^T i = s, \qquad (2)$$

$$L\frac{\mathrm{d}i}{\mathrm{d}t} - \nabla v = 0. \qquad (3)$$

$v = (v_1,\ v_2,\ v_3,\ v_4)^T$ voltage.
$i = (i_1,\ i_2,\ i_3)^T$ current.
s : current applied to each node.

$$C = \begin{pmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & C_4 \end{pmatrix}, L = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{pmatrix}$$



Electrical Network.

Generalize conservation laws $\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$.

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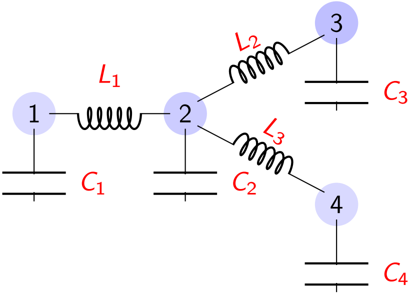

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Graph wave equation

$$C\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \nabla^T L^{-1}\nabla\ v = \frac{\mathrm{d}s}{\mathrm{d}t}. \tag{4}$$

Network wave equation

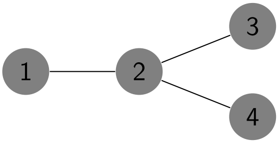
$$C\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \nabla^T L^{-1}\nabla\ v = \frac{\mathrm{d}s}{\mathrm{d}t}. \tag{5}$$

$$\nabla^T L^{-1}\nabla = \begin{pmatrix} L_1^{-1} & -L_1^{-1} & 0 & 0 \\ -L_1^{-1} & L_1^{-1}+L_2^{-1}+L_3^{-1} & -L_2^{-1} & -L_3^{-1} \\ 0 & -L_2^{-1} & L_2^{-1} & 0 \\ 0 & -L_3^{-1} & 0 & L_3^{-1} \end{pmatrix}. \tag{6}$$

$$\frac{d^2 v}{dt^2} = -\Delta v. \tag{7}$$

Graph Laplacian :

$$\Delta = \nabla^T\nabla. \tag{8}$$



Paw graph.

$$\Delta = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Graph Laplacian

Adjacency matrix of $\mathcal{G}(\mathcal{V}, \mathcal{E})$:

$$A_{ij} = \begin{cases} 1 & \text{iff} \quad ij \in \mathcal{E}(\mathcal{G}) \quad (i \sim j) \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

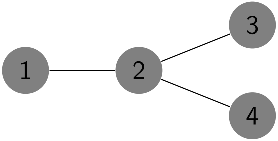
Vertices degree matrix:

$$D_{ii} = d_i = \sum_{j=1}^{N} A_{ij}. \tag{10}$$

Laplacian matrix:

$$\Delta = \nabla^T \nabla = D - A. \tag{11}$$

Example:



Paw graph.

$$\Delta = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Spectrum of graph Laplacian

Laplacian matrix Δ: real, symmetric

$$\Delta\ v^j = \lambda_j\ v^j, \quad j \in \{1, \ldots, N\}\,. \tag{12}$$

- Eigenvalues $\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_N$.
- Eigenvectors v^j : orthonormal.
- $\lambda_1 = 0 \longrightarrow$ Goldstone mode $v^1 = \frac{1}{\sqrt{N}}(1, 1, \ldots, 1)^T$.
- Multiplicity of the eigenvalue 0 $=$ number of connected components of $\mathcal{G}$.

Linear normal modes

$$\frac{d^2 u}{dt^2} = -\Delta u. \tag{13}$$

$$\Delta \ v^j = \lambda_j \ v^j, \quad j \in \{1, \dots, N\}. \tag{14}$$

$$u(t) = \sum_{j=1}^{N} a_j(t) v^j. \tag{15}$$

$$\frac{d^2 a_j}{dt^2} = -\lambda_j a_j. \tag{16}$$

Linear normal modes (linear periodic orbits)

$$u(t) = \left[\frac{\mathrm{d}a_1}{\mathrm{d}t}(0) \ t + a_1(0) \right] v^1 + \sum_{j=2}^{N} \left[a_j(0) \cos\left(\sqrt{\lambda_j} t\right) + \frac{\mathrm{d}a_j}{\mathrm{d}t}(0) \ \frac{\sin\left(\sqrt{\lambda_j} t\right)}{\sqrt{\lambda_j}} \right] v^j. \tag{17}$$

Nonlinear normal modes ?

$$\frac{d^2u}{dt^2} = -\Delta u + N(u). \tag{18}$$

- On-site nonlinearity: **the discrete Φ^4 model**

$$\frac{d^2u}{dt^2} = -\Delta u - u^3. \tag{19}$$

where $u^3 = (u_1^3, u_2^3, \ldots, u_N^3)^T$.

  - Aoki (Phys.Rev.E 94, 2016): chains and cycles.
  - Caputo, Khames, Knippel, Panayotaros (J.Phys.A, 2017): general networks.

- Intersite nonlinearity: **the Fermi-Pasta-Ulam-Tsingou model**

$$\ddot{u}_i = -(\Delta u)_i - \sum_{k\sim i}(u_i - u_k)^3, \quad i \in \{1, \ldots, N\}, \tag{20}$$

  - Bountis, Chechin, Sakhnenko (Int. J. Bif. Chaos, 2011) : chains and cycles.
  - Caputo, Khames, Knippel (2022): general networks.

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Existence of nonlinear periodic orbits

$$\frac{d^2 u}{dt^2} = -\Delta u - u^3. \tag{21}$$

$$\Delta v^j = \lambda_j v^j. \tag{22}$$

Ansatz $u(t) = a_j(t)\ v^j$

$$v^j_m\ \frac{d^2 a_j}{dt^2}\ = [-\lambda_j a_j - a_j^3 (v^j_m)^2]\ v^j_m. \tag{23}$$

Equations (23) are satisfied for $v^j_m = 0$.

$$\frac{d^2 a_j}{dt^2}\ = -\lambda_j a_j - a_j^3 (v^j_m)^2,\ (v^j_m \neq 0). \tag{24}$$

Equations (24) are consistent if and only if

$$(v^j_m)^2 = C,\quad \forall m \in \{1, \ldots, N\}\,. \tag{25}$$

$$C = \frac{1}{N-S},\quad S = \mathrm{card}\left\{s :\ v^j_s = 0\right\}. \tag{26}$$

$$\frac{1}{\sqrt{C}} v^j_m \in \{0, 1, -1\},\quad \forall m \in \{1, \ldots, N\}\,. \tag{27}$$

$$\frac{d^2 a_j}{dt^2}\ = -\lambda_j a_j - C\ a_j^3. \tag{28}$$

Nonlinear periodic orbits

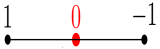
Nonlinear periodic orbits: nonlinear normal modes

nonlinear periodic solutions associated to eigenvectors composed of $\{0, 1, -1\}$.

- Goldstone mode: $\{+1\}$.
- Bivalent modes: $\{+1, -1\}$.
- Trivalent modes: $\{+1, -1, 0\}$.

Definition (Soft node "Caputo, Knippel and Simo 2013")

A node s of a graph is a soft node for an eigenvalue λ of the graph Laplacian if there exists an eigenvector v for this eigenvalue such that $v_s = 0$.



$\lambda = 1.$

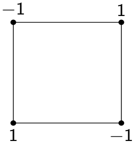
Bivalent and trivalent graphs

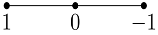
Definition (Bivalent graph)

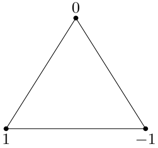
A graph is bivalent if it has an eigenvector of the Laplacian composed of $\{-1, +1\}$. This vecteur is called bivalent.

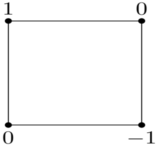
Definition (Trivalent graph)

A graph is trivalent if it has an eigenvector of the Laplacian composed of $\{-1, +1, 0\}$. This vecteur is called trivalent.









Caputo, Khames, Knippel (Discrete Applied Mathematics, 2019) : characterization of graphs having bivalent and trivalent eigenvectors

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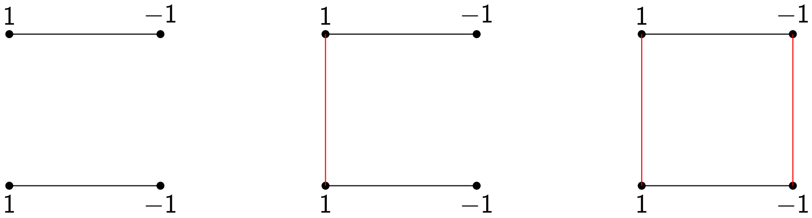
Transformations of graphs

Theorem (Link between two equal nodes, Merris 1998)

Let v be an eigenvector of $\Delta(\mathcal{G})$ affording an eigenvalue λ. If $v_i = v_j$, then v is an eigenvector of $\Delta(\mathcal{G}')$ affording the eigenvalue λ, where $\mathcal{G}'$ is the graph obtained from $\mathcal{G}$ by deleting or adding the edge ij depending on whether or not ij is an edge of $\mathcal{G}$.

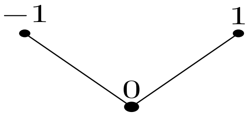


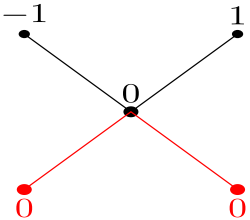
$\lambda = 2.$ $\lambda = 2.$ $\lambda = 2.$

Transformations of graphs

Theorem (Principle of reduction and extension, Merris 1998)

For a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ fix a nonempty subset $\mathcal{W}$ of $\mathcal{V}$. Delete all the vertices in $\mathcal{V}\backslash\mathcal{W}$ that are adjacent in $\mathcal{G}$ to no vertex of $\mathcal{W}$. Remove any remaining edges that are incident with no vertex of $\mathcal{W}$. Suppose v is an eigenvector of the Laplacian of the reduced graph $\mathcal{G}\{\mathcal{W}\}$ that affords λ and is supported by $\mathcal{W}$ in the sense that if $v_i \neq 0$, then $i \in \mathcal{W}$. Then the extension v' with $v'_j = v_j$ for $j \in \mathcal{W}$ and $v'_j = 0$ otherwise is an eigenvector of $\Delta(\mathcal{G})$ affording λ.



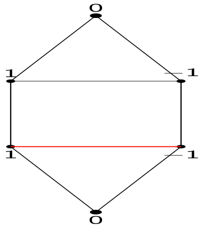


$\lambda = 1.$

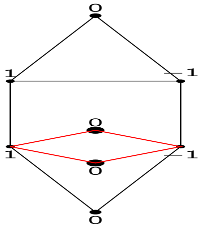
Transformations of graphs

Theorem (Replace an edge by a square "Caputo, Khames, Knippel (2019)")

Let v be an eigenvector of $\Delta(\mathcal{G})$ affording an eigenvalue λ. Let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by deleting an edge ij such that $v_i = -v_j$ and adding two soft nodes $k, l \in \mathcal{V}(\mathcal{G}')$ for the extension v' of v (i.e. $v'_m = v_m$ for $m \in \mathcal{V}(\mathcal{G})$ and $v'_k = v'_l = 0$) and four edges ik, kj, il, lj.
Then, v' is an eigenvector of $\Delta(\mathcal{G}')$ for the eigenvalue λ.



$\lambda = 3.$



$\lambda = 3.$

Characterization of bivalent graphs

Definition (Regular graph)

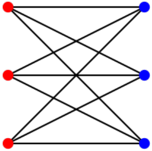
A graph is d-regular if each node of the graph has the degree d.

Definition (Bipartite graph)

a graph whose vertices can be partitioned into two different independent sets so that no two vertices within the same set are adjacent.

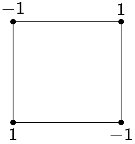
Theorem (Bivalent graphs "Caputo, Khames, Knippel (2019)")

The bivalent graphs are the regular bipartite graphs and their extensions obtained by adding edges between nodes with the same value for a bivalent eigenvector.

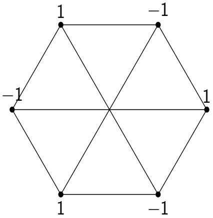


Examples of bivalent graphs

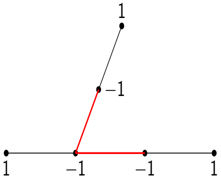
$(d = 1)$-regular.

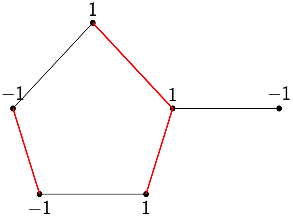


$(d = 2)$-regular.



$(d = 3)$-regular.

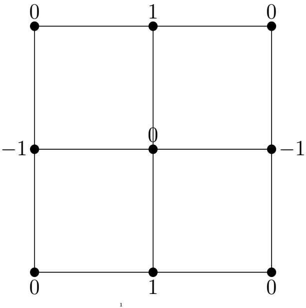




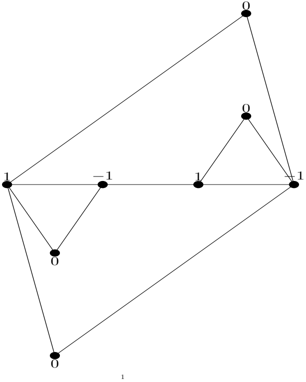
Soft regular graph

Definition (Soft regular graph)

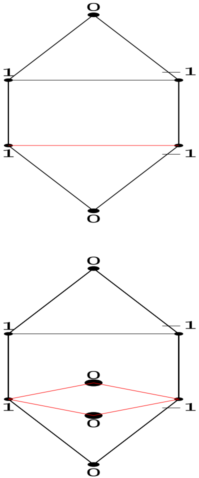
A graph is d-soft regular if there exist an eigenvector v of the graph Laplacian such that the non-soft nodes of v each have degree d.



Soft regular



non-soft regular



Characterization of trivalent graphs

Theorem (Trivalent graphs "Caputo, Khames, Knippel (2019)")

Trivalent graphs are obtained from soft regular graphs by applying on the same trivalent eigenvector the transformations :

- *add link between two equal nodes,*
- *principle of reduction and extension*
- *replace an edge by a soft square.*

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Intersite nonlinearity: nonlinear normal modes

$$\ddot{u}_i = -(\Delta u)_i - \sum_{k \sim i} (u_i - u_k)^3. \tag{29}$$

$$\Delta v^j = \lambda_j v^j. \tag{30}$$

Ansatz

$$u(t) = a_j(t)\ v^j, \tag{31}$$

$$\ddot{a}_j v_i^j = -\lambda_j a_j v_i^j - a_j^3 \sum_{k \sim i} \left(v_i^j - v_k^j\right)^3. \tag{32}$$

(i) if $v_i^j = 0$ (soft nodes), then we have

$$\sum_{k \sim i} \left(v_k^j\right)^3 = 0. \tag{33}$$

(ii) if $v_i^j \neq 0$ then

$$\ddot{a}_j = -\lambda_j a_j - \left[\frac{1}{v_i^j} \sum_{k \sim i} \left(v_i^j - v_k^j\right)^3\right] a_j^3. \tag{34}$$

$$\frac{1}{v_i^j} \sum_{k \sim i} \left(v_i^j - v_k^j\right)^3 = \gamma_j, \quad \forall i \in \{1, \dots, N\} \tag{35}$$

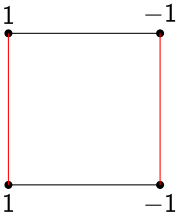
Bivalent graphs yielding NNM

$$\gamma_j = \frac{1}{v_i^j} \sum_{k \sim i} \left(v_i^j - v_k^j \right)^3 \tag{36}$$

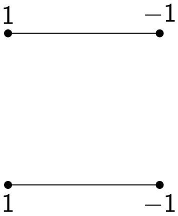
Bivalent graph $\xrightarrow[\text{Removing links between equal nodes}]{}$ regular bipartite graph

$$\gamma_j = 2^3 d_j, \tag{37}$$

independant of the vertex i.



$\lambda = 2.$

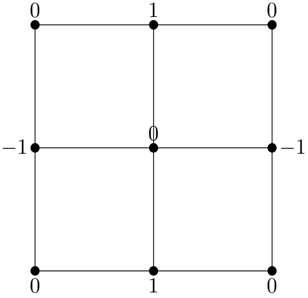


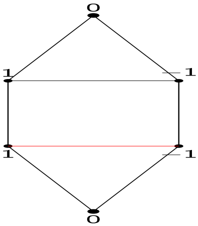
$\lambda = 2.$

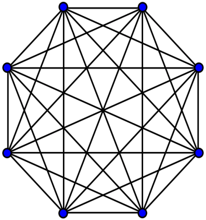
Trivalent-soft-regular graphs yielding NNM

For nonsoft nodes

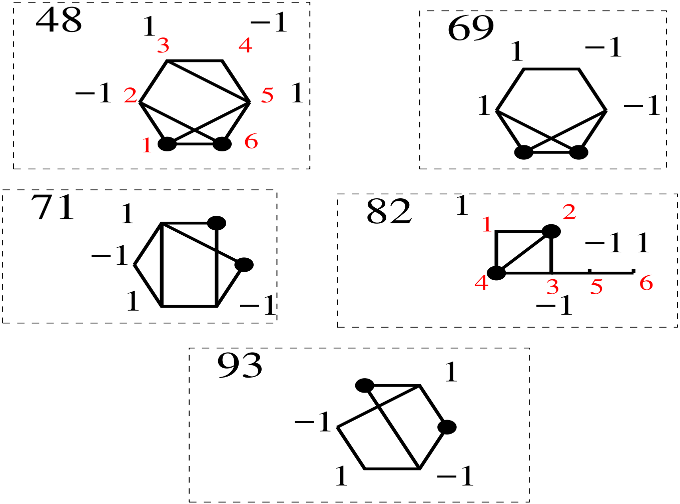
$$\gamma_j = \frac{1}{v_i^j} \sum_{k \sim i} \left(v_i^j - v_k^j \right)^3 = 8d_j - 7s_j \tag{38}$$







Trivalent graphs not yielding NNM

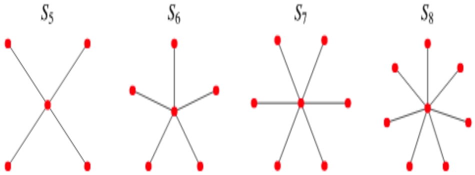


Bipartite complete graph

Definition (Complete bipartite graph $K_{n,N-n}$)

A complete bipartite graph $K_{n,N-n}$ is such that every vertex of the set $\{1, \ldots n\}$ is connected to every vertex of the set $\{n+1, \ldots N\}$.

$$\gamma_j = \frac{1}{v_i^j} \sum_{k \sim i} \left(v_i^j - v_k^j \right)^3 = N^3 \qquad (39)$$



Star graphs $K_{1,N-1}$

Conclusions

- Arbitrary networks with cubic couplings : the FPUT model.
- Graphs yielding nonlinear normal modes
  - On-site nonlinearity: bivalent, trivalent graphs
  - Intersite nonlinearity: bivalent, trivalent soft regular, complete bipartite graphs.
- Characterization of bivalent and trivalent graphs.